Fractional order modeling and identification for heat transfer in lungs

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1. Introduction
2. Thermal Modelling
3. Fractional-order system identification
4. Conclusions and perspectives

Introduction (1/2)

- During cardiac surgery, it is usually required to use extracorporeal circulation (ECC)
- A heart-lung machine is then connected to the patient's circulatory system
- Hypothermia and oxygen consumption are slowed down by means of hypothermia

Introduction (2/2)

- Thermal modelling for transient temperature response in longs : fractional-order models.
  - Heat equation : circuit models.
  - Physiological scenario.
- Fractional-order system identification.
  - Recursive identification.
  - Real-time identification of continuous-time models.

Mathematical background (1/3)

- Grünwald-Letnikov's fractional derivative definition is well suited for implementation :

$$ {}_0\mathbf{D}^{\alpha} f(t) \approx \frac{1}{T_s^{\alpha}} \sum_{j=0}^{\lfloor \frac{t}{T_s} \rfloor} (-1)^j \binom{\alpha}{j} f(t - jT_s) \quad , \alpha \in \mathscr{R}^+ $$

- Newton's generalized binomial :

$$ \binom{\alpha}{j} = \frac{\Gamma(\alpha+1)}{\Gamma(j+1)\Gamma(\alpha-j+1)}. $$

- On Laplace domain with null initial conditions :

$$ \mathscr{L}\{\mathbf{D}^{\alpha} f(t)\} = s^{\alpha} F(s). $$

Mathematical background (2/3)

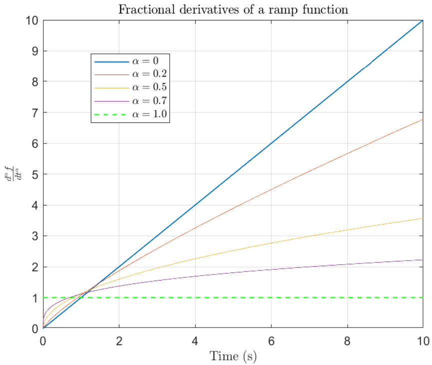

Figure 1 – GL fractional derivative of a ramp function

Mathematical background (3/3)

- Fractional order model of a SISO plant :

$$y(t) + a_1 \mathbf{D}^{\alpha_1} y(t) + \dots a_{m_A} \mathbf{D}^{\alpha_{m_A}} = b_0 \mathbf{D}^{\beta_0} u(t) + b_1 \mathbf{D}^{\beta_1} u(t) + \dots b_{m_B} \mathbf{D}^{\beta_{m_B}} u(t)$$

- Fractional order transfer function :

$$G(s) = \frac{B(s)}{A(s)} = \frac{\sum_{i=0}^{m_B} b_i s^{\beta_i}}{1 + \sum_{j=1}^{m_A} a_i s^{\alpha_i}}.$$

- Commensurate transfer function :

$$G(s) = \frac{B(s)}{A(s)} = \frac{\sum_{i=0}^{m_B} b_i s^{i\nu}}{1 + \sum_{j=1}^{m_A} a_i s^{j\nu}}.$$

Thermal two-port network formalism (1/3) [1]

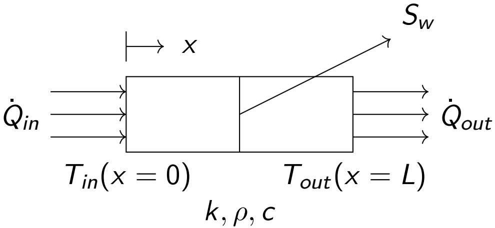

Figure 1 – 1D thermal system

- Heat equation :

$$\rho c \frac{\partial T}{\partial t} = \nabla^2 T(x, y, z, t),$$

- For a 1D case and by taking the Laplace transform :

$$sT(x, s) = \frac{k}{\rho c} \frac{\partial^2 T(x, s)}{\partial x^2}.$$

1. D. Maillet et al. *Thermal Quadrupoles : Solving the Heat Equation through Integral Transforms*. John Wiley et sons, 2000.

Thermal two-port network formalism (2/3)

- In and out heat flows are expressed as :

$$\dot{Q}_{in}(s) = -kS_w \left.\frac{\partial T(x,s)}{\partial x}\right|_{x=0}$$
$$\dot{Q}_{out}(s) = -kS_w \left.\frac{\partial T(x,s)}{\partial x}\right|_{x=L}$$

- Under matrix form :

$$\begin{bmatrix} T_{in}(s) \\ \dot{Q}_{in}(s) \end{bmatrix} = \boldsymbol{M} \begin{bmatrix} T_{out}(s) \\ \dot{Q}_{out}(s) \end{bmatrix}$$

- Transmission matrix :

$$\boldsymbol{M} = \begin{bmatrix} \cosh(\delta L) & \frac{1}{kS_w\delta}\sinh(\delta L) \\ kS_w\delta\sinh(\delta L) & \cosh(\delta L) \end{bmatrix}$$

with $\delta = \sqrt{\frac{s}{a}}$ and $a = \frac{k}{\rho c}$.

Thermal two-port network formalism (3/3)

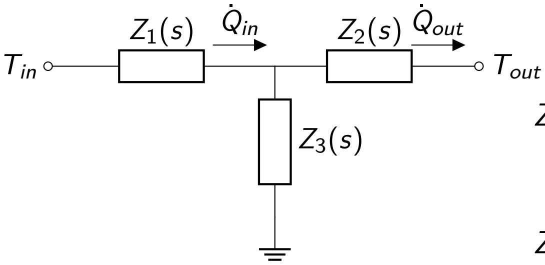

Figure 1 – Thermal system equivalent circuit

$$Z_1(s) = \frac{1}{kS_w\delta}[\coth(\delta L) - \operatorname{csch}(\delta L)] = Z_2(s)$$

$$Z_3(s) = \frac{1}{kS_w\delta}\operatorname{csch}(\delta L)$$

Thermal impedance approximation : $Z_1(s)$ and $Z_2(s)$

- Low-frequency behavior :

$$\lim_{\omega \to 0} Z_1(j\omega) = \frac{L}{2kS_w} = R$$

- High-frequency behavior :

$$Z_{1-HF}(s) = \frac{1}{C_s s^{0.5}}$$

- Asymptotic approximation :

$$Z_{1-asymp}(s) = \frac{R}{1 + RC_s\sqrt{s}}$$

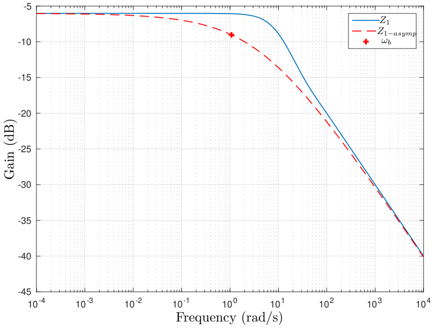

Figure 1 – Z_1 true gain and asymptotic approximation

Thermal impedance approximation : $Z_1(s)$ and $Z_2(s)$

- Asymptotic approximation may lead to significant error around mid-band frequencies.
- Zeros and poles cells are added :

$$Z_{1-pz}(s) = \frac{R}{1 + RC\sqrt{s}} \prod_{i}^{N_{cells}} \frac{1 + \frac{s}{z_i}}{1 + \frac{s}{p_i}}$$

- Parameter vector :

$$\theta = [\boldsymbol{p} \quad \boldsymbol{z}]$$

- Issue : θ may have an important dimension

Thermal impedance approximation : $Z_1(s)$ and $Z_2(s)$

- Alternative proposition :

$$Z_{1-BW}(s) = \frac{d}{s^{\alpha+\beta} + as^{\alpha} + bs^{\beta} + c}$$

- By considering asymptotic behavior :

$$\alpha + \beta = 0.5, \quad c = \frac{1}{RC}, \quad d = \frac{1}{C}$$

- The new parameter vector :

$$\theta = [\alpha \quad a \quad b].$$

Thermal impedance approximation : $Z_1(s)$ and $Z_2(s)$

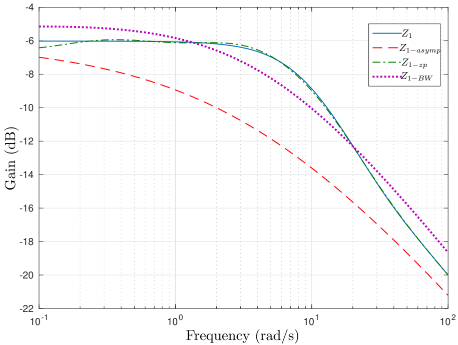

Figure 1 – Z_1 true gain and approximations

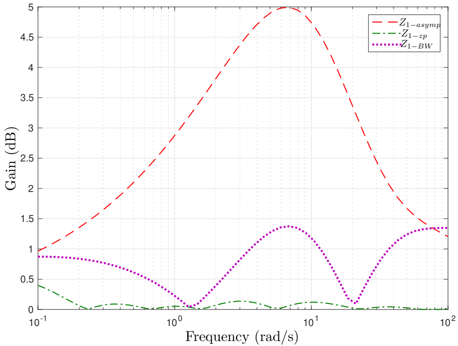

Figure 2 – Error for Z_1 approximations

Thermal impedance approximation : $Z_3(s)$

- Low-frequency behavior :

$$Z_{3-cap}(s) = \frac{1}{C_t s}$$

- High-frequency approximation :

$$\lim_{\omega \to \infty} \arg |Z_3(j\omega)| = -\infty$$

- For a given frequency band, an approximation has the following form :

$$Z_{3-HF}(s) = \frac{1}{C_t s} H_{filter}(s)$$

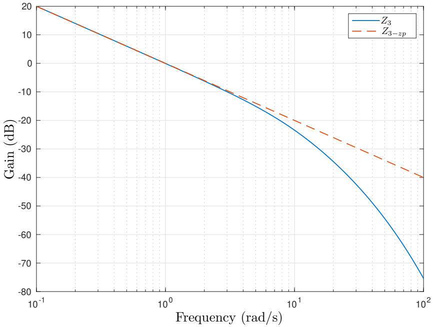

Figure 1 – Z_3 true gain and capacitance approximation

Thermal impedance approximation : $Z_3(s)$

- Fractional-order slope :

$$Z_{3-frac}(s) = \frac{1}{C_t s}\left[\frac{1}{1+(\tau s)^{\phi}}\right]$$

- Parameter vector :

$$\theta_{frac-slope} = [\tau \quad \phi]$$

- Multiple fractional-order slopes :

$$Z_{3-mult-frac}(s) = \frac{1}{C_t s}\prod_{i=1}^{N}\frac{1}{1+(\tau_i s)^{\nu}}$$

- Integer order poles :

$$Z_{3-rec-poles}(s) = \frac{1}{C_t s}\prod_{i=1}^{\tilde{N}}\frac{1}{1+\tau_i s}$$

- Relationship between poles :

$$\tau_{i+1} = \tau_1^{i\gamma}$$

- Limited parameter vector :

$$\theta_{red} = [\tau_1 \quad \gamma]$$

Thermal impedance approximation : $Z_3(s)$

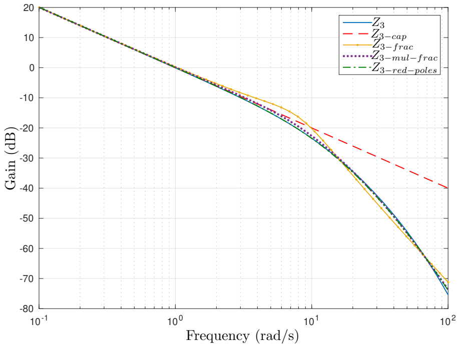

Figure 1 – Z_3 true gain and approximations

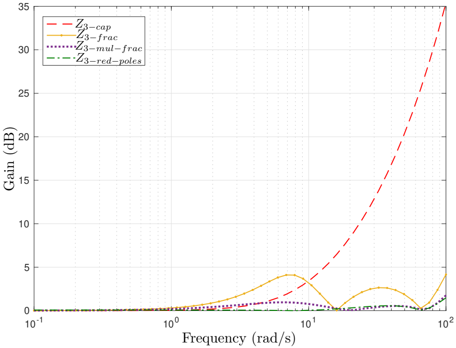

Figure 2 – Error for Z_3 approximations

Bio-heat two-port network (1/2)

- Bio-heat equation :

$$\rho c \frac{\partial T}{\partial t} = k\nabla^2 T + \rho_b c_b \omega_b (T_a - T) + Q_{met} + Q_{ext}$$

- Variable change :

$$\tilde{T} = T - T_{body}$$

- By supposing $T_a \approx T_{body}$ and neglecting q_{met} and q_{ext} :

$$\rho c \frac{\partial \tilde{T}}{\partial t} = k\nabla^2 \tilde{T} - \rho_b c_b \omega_b \tilde{T}$$

- On Laplace domain :

$$\rho c s \tilde{T} = k \frac{\partial^2 \tilde{T}}{\partial x^2} - \rho_b c_b \omega_b \tilde{T}$$

Bio-heat two-port network (2/2)

- We consider :

$$a = \frac{k}{\rho c}, \quad h_{blood} = \frac{\rho_b c_b \omega_b}{k}$$

- Bio-heat equation leads to :

$$\frac{\partial^2 \tilde{T}(z,s)}{\partial z^2} = [\frac{s}{a} + h_{blood}]\tilde{T}(z,s)$$

- A very similar thermal two-port network is found, with one difference :

$$\delta = \sqrt{\frac{s}{a} + h_{blood}}$$

Lung structure and geometrical relationships [2]

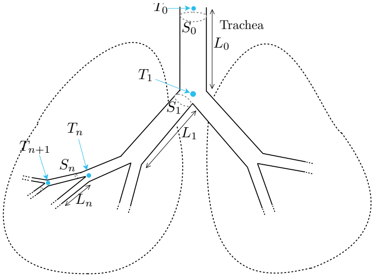

Figure 1 – Lung scheme

- Each branch n has length L_n and cross section S_n.
- Murray's law :

$$\frac{S_{n+1}}{S_n} = 2^{-\frac{2}{3}}$$

$$\frac{L_{n+1}}{L_n} = 2^{-\frac{1}{3}}$$

2. Fujio Kuwahara et al. "A Porous Media Approach for Bifurcating Flow and Mass Transfer in a Human Lung". In : *Journal of Heat Transfer* 131.101013 (juill. 2009). ISSN : 0022-1481. DOI : 10.1115/1.3180699. URL : https://doi.org/10.1115/1.3180699 (visité le 16/03/2021).

Global circuit model

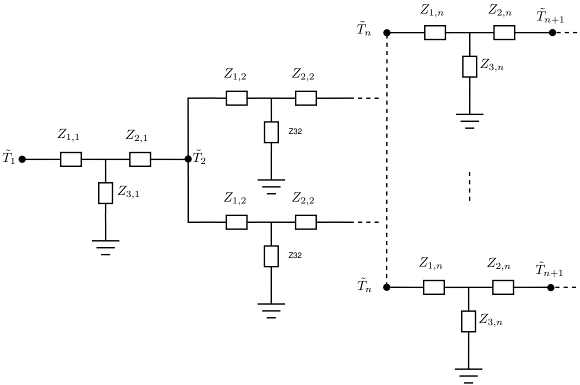

Figure 1 – Global circuit model for the lung

Equivalent thermal impedance

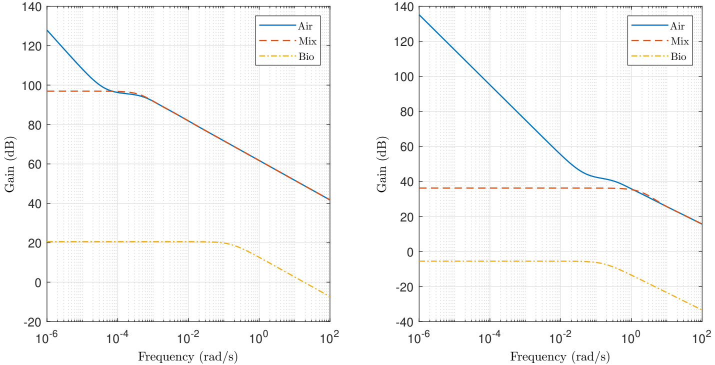

Figure 1 – Input equivalent impedance at $n = 1$ and $n = 13$

1. Introduction
2. Thermal Modelling
3. **Fractional-order system identification**
4. Conclusions and perspectives

System identification general problem

- Consider an input $u(t)$ and output $y^*(t)$ with a sampling period T_s going from $t = 0$ to $t = T_f$.
- Sensor imperfections as well as experience errors are considered by adding a noise signal $p(t)$ to ideal noise-free signal $y(t)$:

$$y^*(t) = y(t) + p(t)$$

- Supposing input and noise-free output are related by a transfer function :

$$G(s) = \frac{Y(s)}{U(s)} = \frac{\sum_{i=0}^{m_B} b_i s^{\beta_i}}{1 + \sum_{j=1}^{m_A} a_i s^{\alpha_i}}.$$

Recursive Least Squares with State Variable Filter (RLSSVF) (1/3)

- Linear relationship between parameter vector θ and noise-free output $y(t)$.

$$y(t) = \phi^*(t)^T\theta$$

- Matrix ϕ^* :

$$\phi^*(t)^T = [\mathbf{D}^{\beta_0}u(t)\dots\mathbf{D}^{\beta_{m_B}}u(t) \quad -\mathbf{D}^{\alpha_1}y^*(t)\dots -\mathbf{D}^{\alpha_{m_A}}y^*(t)]$$

- The error :

$$\epsilon(t) = y^*(t) - \phi^*(t)^T\theta.$$

- In order to limit noise impact, a Poisson filter is used :

$$F(s) = \frac{1}{\left[\left(\frac{s}{\omega_f}\right)^n + 1\right]^{N_f}}.$$

Recursive Least Squares with State Variable Filter (RLSSVF) (2/3)

- Filtered input $u_f(t)$ and output $y_f^*(t)$:

$$u_f(t) = \{\mathscr{L}^{-1} F(s)\} * u(t)$$
$$y_f^*(t) = \{\mathscr{L}^{-1} F(s)\} * y^*(t)$$

- Filtered error $\epsilon_f(t)$:

$$\epsilon_f(t) = y_f^*(t) - \phi_f^*(t)^T \theta$$

- Matrix $\phi_f^*(t)$:

$$\phi_f^*(t)^T = [\mathbf{D}^{\beta_0} u_f(t) \dots \mathbf{D}^{\beta_{m_B}} u_f(t) \quad - \mathbf{D}^{\alpha_1} y_f^*(t) \dots - \mathbf{D}^{\alpha_{m_A}} y_f^*(t)]$$

Recursive Least Squares with State Variable Filter (RLSSVF) (3/3)

- Correction term :

$$L(k) = \frac{F(k-1)\phi_f^*(k)}{\lambda + \phi_f^*(k)^T F(k-1)\phi_f^*(k)}$$

- Parameter update :

$$\hat{\theta}(k) = \hat{\theta}(k-1) + L(k)\left[y_f^*(k) - \phi_f^*(k)^T \hat{\theta}(k-1)\right]$$

- Matrix F update :

$$F(k) = \frac{1}{\lambda}\left[I - L(k)\phi_f^*(k)^T\right] F(k-1)$$

Prediction Error Method (PEM) (1/2)

- Quadratic criterion is to be minimized :

$$J(\theta) = \frac{1}{2} \sum_{k=1}^{t} \epsilon(k,\theta)^2$$

- Criterion gradient :

$$\frac{\partial J(\theta)}{\partial \theta} = -\sum_{k=1}^{t} \psi(k,\theta)\epsilon(k,\theta)$$

- Function ψ :

$$\psi(k,\theta) = -\frac{\partial \epsilon(k,\theta)}{\partial \theta}$$

Prediction Error Method (PEM) (2/2)

- Estimation error :

$$\epsilon(k) = y^*(k) - \hat{y}(k)$$

- R update :

$$R(k) = R(k-1) + \gamma \left[\psi(k,\theta)\psi^T(k,\theta) - R(k-1)\right]$$

- Parameter update :

$$\hat{\theta}(k) = \hat{\theta}(k-1) + \gamma R^{-1}(k)\psi(k,\theta)\epsilon(k)$$

Long Memory Prediction Error Method (LMRPEM) (1/2)

- Fractional-order operatos have non-local nature. This suggests using more information on each iteration to correct parameter vector.
- Extended error function :

$$\tilde{\epsilon}(k) = [\epsilon(0)\,\epsilon(T_s)\,\epsilon(2T_s)\ldots\epsilon(kT_s)]^T$$

- Corresponding sensitivity functions :

$$\psi_{ex}(K,\theta) = -\begin{bmatrix} \frac{\partial\epsilon(0)}{\partial b_0} & \frac{\partial\epsilon(T_s)}{\partial b_0} & \frac{\partial\epsilon(2T_s)}{\partial b_0} & \cdots & \frac{\partial\epsilon(kT_s)}{\partial b_0} \\ \vdots & & \vdots & \ddots & \vdots \\ \frac{\partial\epsilon(0)}{\partial a_{m_A}} & \frac{\partial\epsilon(T_s)}{\partial a_{m_A}} & \frac{\partial\epsilon(2T_s)}{\partial a_{m_A}} & \cdots & \frac{\partial\epsilon(kT_s)}{\partial a_{m_A}} \end{bmatrix}$$

Long Memory Prediction Error Method (LMRPEM) (2/2)

- Estimation error :

$$\tilde{\epsilon}(kT_s) = \tilde{Y}^*(kT_s) - \tilde{Y}(kT_s)$$

- R update

$$R(k) = R(k-1) + \gamma\left[\psi_{ex}(kT_s,\theta)\psi_{ex}^T(kT_s,\theta) - R(k-1)\right]$$

- Parameter update :

$$\hat{\theta}(k) = \hat{\theta}(k-1) + \gamma R^{-1}(k)\psi_{ex}(kT_s,\theta)\epsilon(kT_s)$$

Simulation example

- System true model :

$$G(s) = \frac{b_0}{1 + a_1 s^{\nu} + a_2 s^{2\nu}}$$

- Parameter values $b_0 = 1$, $a_1 = 1$, $a_2 = 2$ and $\nu = 0.5$.
- PRBS input going from -5 à 5 with 1000 points.
- Sampling period $T_s = 0.1\,s$
- Initial conditions $\hat{b}_0 = 1$ $\hat{a}_1 = 0.5$ and $\hat{a}_2 = 2.5$.
- Poisson filter parameters $\omega_f = 1$ rad/s, $n = 1$ and $N_f = 2$.

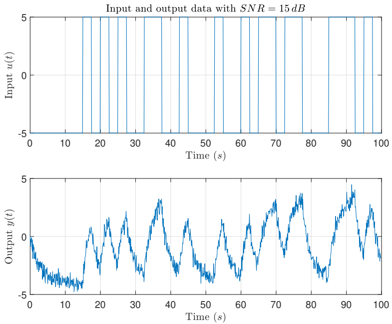

Figure 1 – Input and output with $SNR = 15\,dB$

RLSSVF

SNR	a_2	a_1	b_0
True	2.00	1.00	1.00
20	2.023 ± 0.021	1.060 ± 0.031	1.000 ± 0.008
15	2.002 ± 0.034	1.017 ± 0.031	0.987 ± 0.013
10	1.946 ± 0.065	0.909 ± 0.095	0.946 ± 0.025
5	1.854 ± 0.098	0.595 ± 0.124	0.856 ± 0.034
3	1.736 ± 0.116	0.400 ± 0.166	0.785 ± 0.045

Table 1 – Mean values and standard deviations with RLSSVF for a 100 run Monte Carlo simulation

- Small variations, slow stabilisation ☺
- Sensitive to noise, important bias ☹

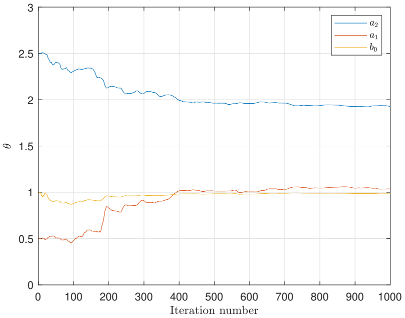

Figure 1 – RLSSVF estimation for $SNR = 15\,dB$

RPEM

SNR	a_2	a_1	b_0
True	2.00	1.00	1.00
20	2.008 ± 0.042	1.009 ± 0.091	1.004 ± 0.028
15	1.995 ± 0.083	1.022 ± 0.179	1.005 ± 0.059
10	2.003 ± 0.142	1.025 ± 0.303	1.009 ± 0.099
5	2.026 ± 0.267	1.250 ± 0.502	1.078 ± 0.159
3	1.989 ± 0.390	1.273 ± 0.695	1.082 ± 0.205

Table 2 – Mean values and standard deviations with RPEM for a 100 run Monte Carlo simulation

- Many oscillations for parameter oscillation, slower transient ☹
- More robust to noise, no bias. ☺

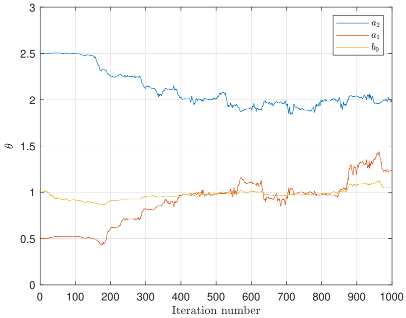

Figure 2 – RPEM estimation for $SNR = 15\,dB$

LMRPEM

SNR	a_2	a_1	b_0
True	2.00	1.00	1.00
20	1.998 ± 0.024	1.001 ± 0.030	1.000 ± 0.007
15	1.993 ± 0.044	1.003 ± 0.060	0.999 ± 0.015
10	2.010 ± 0.073	0.9805 ± 0.092	0.995 ± 0.024
5	1.982 ± 0.149	1.049 ± 0.180	1.009 ± 0.040
3	1.974 ± 0.116	1.023 ± 0.213	1.002 ± 0.053

Table 3 – Mean values and standard deviations with LMRPEM for a 100 run Monte Carlo simulation

- Slow transient, no oscillation. ☹
- Robust to noise, no bias. ☺

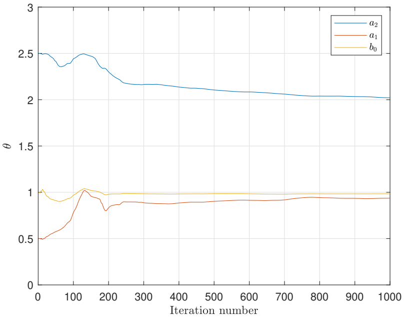

Figure 3 – LMRPEM estimation for $SNR = 15\,dB$

Frequency response

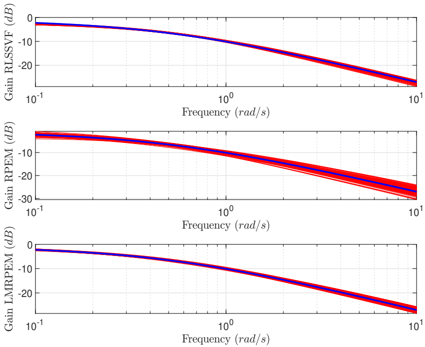

Figure 4 – Bode plots for Monte Carlo simulations with $SNR = 5\,dB$

Commensurate order influence

- True order $\nu = 0.5$.
- A wrong order may lead to important estimation errors.

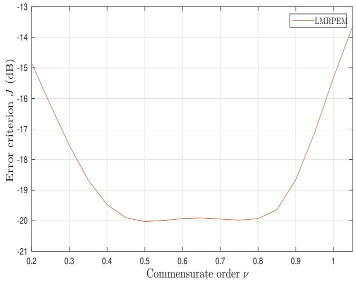

Figure 5 – Criterion J vs commensurate order ν ($SNR = 20\,dB$)

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Conclusions

- Thermal impedance approximations for 1D heat conduction
- Inclusion of blood perfusion influence on heat transfer for physiological scenarios
- Simple thermal equivalent impedance from lung global circuit model
- Coefficient identification of a fractional-order transfer function without bias
- Algorithms were adapted to be recursive

Perspectives

- Thermal impedance parameter analysis : optimization by using phase
- Commensurate order ν recursive estimation [3]
- Long-memory problem : real-time implementation issue [4]

3. Stéphane Victor et al. “Long Memory Recursive Prediction Error Method for Identification of Continuous-time Fractional Models”. In : (jan. 2022). DOI : 10.21203/rs.3.rs-1272889/v1.
4. Jean-François Duhé et al. “Fractional Derivative Truncated approximation for real-time system identification”. In : *Communications in Nonlinear Science and Numerical Simulation* (2022).

Questions ?