Partial moments in system identification, a tool for initializing of output error algorithms

An application to population dynamics model estimation

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Part 1:

Partial moments in system identification

Principle, properties, some expansions

Part 2:

New models of population dynamics

A study of the French Yellowhammer population

State-of-the-art in population dynamics

A PDE model with variable parameters

Methodology for parametric estimation

Initialisation by the partial moment approach

Problem formulation

Let us consider a system described by the ordinary differential equation[1] of order $n_a \geq n_b$:

$$y_0^{(n_a)}(t) + \sum_{i=0}^{n_a-1} a_i y_0^{(i)}(t) = \sum_{i=0}^{n_b} b_i u^{(i)}(t) \qquad (1)$$

The goal is to estimate the parameters a_i and b_i by using a linear least squares from input-output discrete measures $\{u(k\Delta t), y(k\Delta t)\}_{k=0,\ldots,N_t}$ where $y(k\Delta t) = y_0(k\Delta t) + v(k\Delta t)$ is the noisy output.

Partial moments approach

A repeated integration of the ODE (1) to remove the derivatives $u^{(i)}(t)$ and $y_0^{(i)}(t)$

[1] $y_0(t)$ is the noise free output of the system and $u(t)$, the input

Let us define the n-th order partial moment of a $\mathcal{L}^2$ function $g(t)$

$$\mathcal{M}_n^g(T) = \int_0^T \frac{t^n}{n!} g(t) dt$$

A simple case

Consider a first-order system defined by the following ODE:

$$\frac{dy_0(t)}{dt} = -a_0 y_0(t) + b_0 u(t) \quad (2)$$

By applying a first order partial moment to (2):

$$\int_0^T t \left(\frac{dy_0(t)}{dt} = -a_0 y_0(t) + b_0 u(t) \right) dt \quad (3)$$

A simple case

Using integration by parts, Eq. (3) yields the following noise free output formulation:

$$y_0(T) = -a_0 \frac{\mathcal{M}_1^{y_0}(T)}{T} + b_0 \frac{\mathcal{M}_1^{u}(T)}{T} + \frac{\mathcal{M}_0^{y_0}(T)}{T}$$

In practice, considering the noisy output measure, an approximation of y_0 is given at the instant T as follows:

$$\bar{y}(T) = -a_0 \frac{\mathcal{M}_1^{y}(T)}{T} + b_0 \frac{\mathcal{M}_1^{u}(T)}{T} + \frac{\mathcal{M}_0^{y}(T)}{T} \quad (4)$$

which presents a variance depending on noise level and integration interval $[0, T]$.
⇒ This formulation presents a minimal variance for an optimal interval T_{opt}.

Let us define the *n-th order reinitialized partial moment* of a $\mathcal{L}^2$ function $g(t)$

$$M_n^g(t) = \int_0^{\widehat{T}} \tau^n g(t - \widehat{T} + \tau) d\tau$$

A simple case

By substituting $\mathcal{M}_n^{\bullet}(T)$ for $M_n^{\bullet}(t)$ in Eq. (4), an estimate of the noise free output y_0 is given $\forall t > \widehat{T}$ by:

$$\widehat{y}(t) = -\widehat{a}_0 \frac{M_1^y(t)}{\widehat{T}} + \widehat{b}_0 \frac{M_1^u(t)}{\widehat{T}} + \frac{M_0^y(t)}{\widehat{T}} \qquad (5)$$

with the property of an optimal filtering of the noise if $\widehat{T} = T_{opt}$.

A simple case

Eq. (5) can be rewritten as follows:

$$\widehat{y}(t) = -\widehat{a}_0(m(t) * y(t)) + \widehat{b}_0(m(t) * u(t)) + \left(\delta(t) - \frac{dm(t)}{dt}\right) * y(t)$$

where $\delta(t)$ is the Dirac function, $*$ is the convolution product and $m(t)$ is a implicit FIR filter defined by

$$m(t) = \begin{cases} \frac{\widehat{T}-t}{\widehat{T}} & \text{if } t \in \left[0, \widehat{T}\right] \\ 0 & \text{else} \end{cases}$$

Properties of the partial moments approach

- A matrix formalism for any order n_a
- An optimal filtering of the noise due to the *reinitialisation* of partial moments and to the implicit FIR filter

$$m(t) = \frac{(\widehat{T} - t)^{n_a} t^{n_a - 1}}{(n_a - 1)!\,\widehat{T}^{n_a}} \quad \text{with} \quad t \in \left[0, \widehat{T}\right] \tag{6}$$

- The design parameter $\widehat{T}$, the reinitialisation interval, which allows to adapt to the type of noise (white or colored)

Properties of the partial moments approach in the discrete-time context

- An optimal filtering of the noise and a discrete-time implicit FIR filter defined by

$$m_j = \frac{(j+1)(j+2)\dots(j+n_a-1)A^{n_a}_{\widehat{\mathcal{T}}-j}}{(n_a-1)!A^{n_a}_{\widehat{\mathcal{T}}}}$$

with $j = 0, \dots, \widehat{\mathcal{T}} - n_a$ and $A^n_j = \frac{j!}{(j-n)!}$

- The implicit FIR filter is close to the ideal filter of Steiglitz-McBride

Algebraic identification

- The algebraic approach introduced by Michel Fliess and Hebertt Sira-Ramirez and the partial moments approach introduced by Jean-Claude Trigeassou yield to the same result

Notice the equivalences between the Laplace domain and the time representation for $n \geq 0$ and null initial conditions

$$\mathcal{L}\left\{(-1)^n t^n g(t)\right\} = \frac{d^n G(s)}{ds^n}$$

$$\mathcal{L}\left\{\int_0^t \int_0^{\tau_1} \cdots \int_0^{\tau_{n-1}} g(\tau_n) d\tau_n \cdots d\tau_2 d\tau_1\right\} = s^{-n} G(s)$$

Some expansions

- Continuous-time LPV model estimation
- PDE model estimation
- Nonlinear system estimation
- Fractional order system estimation

$\Rightarrow$ **A tool for initializing of output error algorithms**

Partial moments in system identification

Contents

- An introduction about moments in identification
- An introductory example
- Partial moments in continuous-time
- Partial moments in discrete-time
- Algebraic identification, a partial moment approach
- Continuous-time subspace based method
- Continuous-time linear parameter varying model
- Multidimensional partial moments
- Nonlinear system estimation
- Fractional order system estimation

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Study of the French Yellowhammer population

A species impacted by climate change and intensive agriculture

New models of population dynamics

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New models of population dynamics
Context & Objectives

Context:
Use the system identification approaches to improve the modelling of impacts on Biodiversity

Objectives:

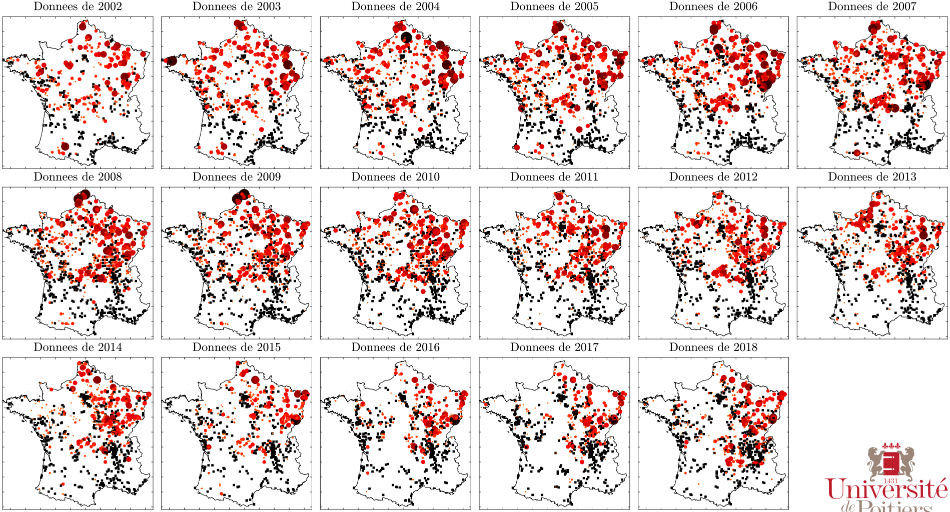
- Propose new models of population dynamics
- Evaluate the impacts of global changes (climate, agriculture,...) on wildlife
- Develop decision support tools for public policies in favour of biodiversity

Naturalist data on common birds

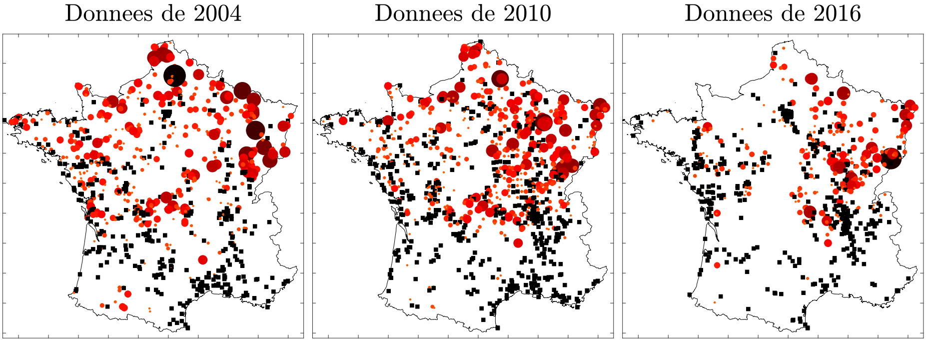
Suivi Temporel des Oiseaux Communs (STOC), this is a counting protocol with approximately 700 observation points spread over France since 2002

One time sample per year

Data on Yellowhammer population



Data on Yellowhammer population

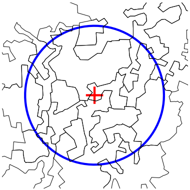


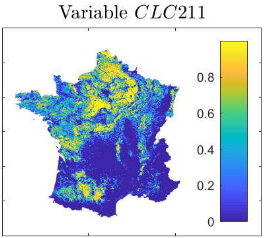
Habitat data from the CORINE Land Cover database

- An inventory of land cover in 44 labels

CLC code	Label
111	Continuous urban fabric
211	Non-irrigated arable land
221	Vineyards
...	...

- Proportion of each label within a radius of 1,5 km
- One time sample every six years



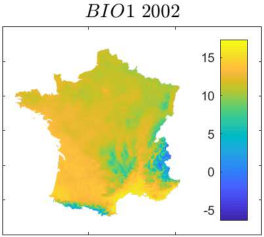


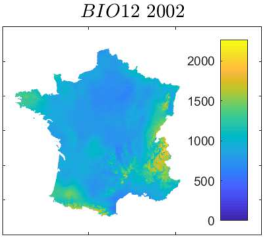
Global climate data from the WorldClim database

- Nineteen standard WorldClim bioclimatic variables

BIO1	Annual Mean Temperature
BIO2	Mean Diurnal Range (Mean of monthly (max temp - min temp))
...	...
BIO12	Annual Precipitation
BIO13	Precipitation of Wettest Month
...	...

- One time sample per year





New models of population dynamics

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In population dynamics, there are two main classes of models

- Statistical models (probabilistic)
  Examples: GLM, Maxent, Ecological niche model,...
- Deterministic models based on ordinary differential equations

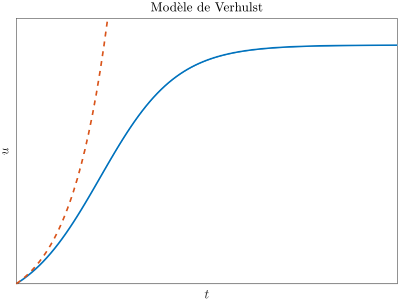
Examples:

- Exponential growth model

$$\frac{du(t)}{dt} = ru(t)$$

- Logistic growth model

$$\frac{du(t)}{dt} = ru(t)\left(1 - \frac{u(t)}{K}\right)$$



Estimation of a GLM statistical model

- Poisson probability distribution and link function *log*
- Model with Intercep, linear and quadratic terms
- Explicative variables averaged over 5 years
- Selection of explicative variables by using AIC criterion among the 44 CLC labels and the 19 bioclimatic variables

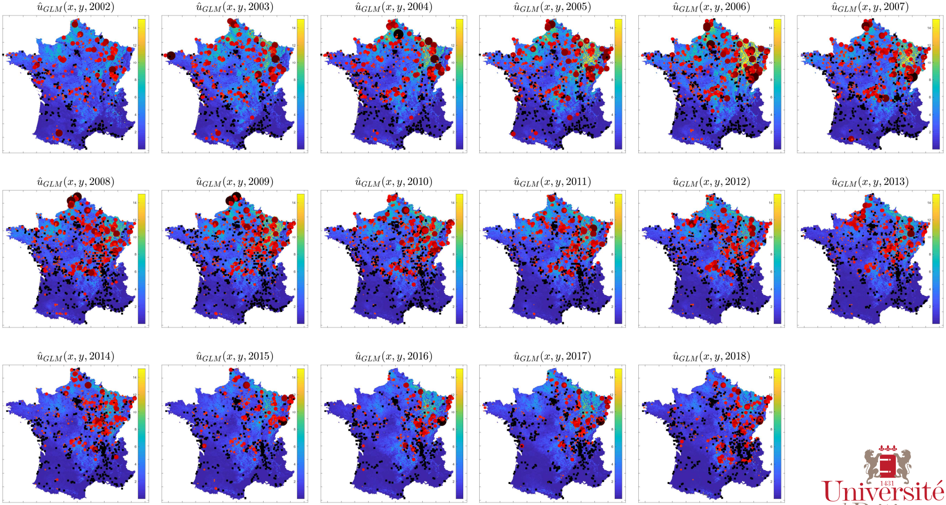
Estimated GLM model (Results)

- GLM model with 80 parameters β_i with $i = 0, \ldots, 79$

$$\hat{u}_{GLM}(x, y, t) = e^{\left(\beta_0 + \sum_{i=1}^{18} \beta_{i+1} BIO{-}{-}(x,y) + \sum_{i=1}^{14} \beta_{i+19} BIO{-}{-}^2(x,y) + \sum_{i=1}^{25} \beta_{i+33} CLC{-}{-}{-}(x,y) + \sum_{i=1}^{21} \beta_{i+58} CLC{-}{-}{-}^2(x,y)\right)}$$

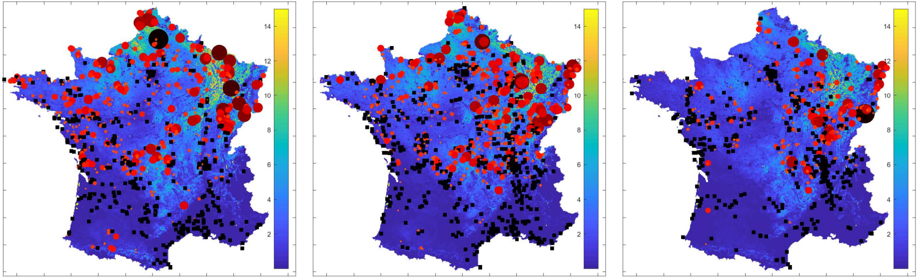
- *Fitting* of 25 %

Output $\hat{u}_{GLM}(x, y, t)$ of the estimated GLM model



Output $\hat{u}_{GLM}(x, y, t)$ of the estimated GLM model

$\hat{u}_{GLM}(x, y, 2004)$ $\hat{u}_{GLM}(x, y, 2010)$ $\hat{u}_{GLM}(x, y, 2016)$



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Deterministic model based on the PDE with variable parameters

$$\frac{\partial u}{\partial t} = \mathrm{div}\left((D_0 + D_1 H)\nabla u(x,y,t)\right) - w_x(1-H)\frac{\partial u}{\partial x} - w_y(1-H)\frac{\partial u}{\partial y}$$
$$+(\beta_{1_0} + \beta_{1_1} H)u(x,y,t) - (\beta_{2_0} + \beta_{2_1} H)u^2(x,y,t)$$

with a scheduling variable $H(x,y,t)$ determined from habitat and bioclimatic data

Parametric estimation under difficult conditions

- Estimation of the parameters of an PDE
- Only 17 time samples
- Poor excitation
- Lack of prior knowledge about parameters

Estimation of parameters by minimization of the quadratic criterion

$$J(\boldsymbol{\theta}) = \sum_{k=0}^{N_t}\sum_{j=0}^{N_y}\sum_{i=0}^{N_x}\left(u(i.dx, j.dy, k.dt) - \hat{u}(i.dx, j.dy, k.dt)\right)^2$$

with

$\boldsymbol{\theta} = [D_0 \;\; D_1 \;\; w_x \;\; w_y \;\; \beta_{1_0} \;\; \beta_{1_1}]^\top$

$N_t = 16$, $N_x = 500$ et $N_y = 500$

u, measured data

$\hat{u}$, simulation or approximation of the PDE for given parameters

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Principle of the Galerkin method:

Time-space separation basic idea

- Search for an approximate solution of the PDE such that

$$\hat{u}(x, y, t) = \sum_{n=1}^{N} a_n(t)\phi_n(x, y)$$

with

$\{\phi_n(x, y)\}_{n=1}^{N}$, an orthonormal function basis

$\{a_n(t)\}_{n=1}^{N}$, the time coefficients to be calculated

- Applying the inner product $\langle \bullet, \phi_\ell \rangle$, the PDE problem is transformed into a system of ODEs with the time coefficient vector

$$\mathbf{a}(t) = \begin{bmatrix} a_1(t) & \dots & a_N(t) \end{bmatrix}^\top$$

- System of ODEs in matrix form

$$\dot{\mathbf{a}}(t) = (\mathbf{\Gamma}_1 + \beta_{1_0}\mathbf{I}_N - 2\mathbf{\Gamma}_2)\ \mathbf{a}(t) - \left(\mathbf{I}_N \otimes \mathbf{a}^\top(t)\right) \mathbf{\Lambda}\ \mathbf{a}(t) + \mathbf{b}_1$$

with

$\otimes$, Kronecker product
$\mathbf{I}_N$, $N \times N$ identity matrix
$\mathbf{\Gamma}_1, \mathbf{\Gamma}_2, \mathbf{\Lambda}, \mathbf{b}_1$, function matrices of the parameters

- with known initial conditions

$$a_\ell(0) = \int (u(x,y,0) - \bar{u}(x,y))\,\phi_\ell(x,y),$$

for $\ell = 1, \ldots, N$ and $\bar{u}(x,y)$, the time average of the population u.

Choice of the orthonormal function basis:
Proper orthogonal decomposition (POD)

- The POD approach leads to empirical orthonormal functions which is the best approximation of $u(x, y, t)$ in the least-squares sense
- Empirical sampled covariance matrix (*snapshots method*)

$$\mathbf{K} = \begin{bmatrix} \alpha_{11} & \cdots & \alpha_{1M} \\ \vdots & \ddots & \vdots \\ \alpha_{M1} & \cdots & \alpha_{MM} \end{bmatrix}$$

with

$$\alpha_{k\ell} = \frac{1}{M} \int \int_{\mathcal{D}} (u(x, y, k\,dt) - \bar{u}(x, y))(u(x, y, \ell\,dt) - \bar{u}(x, y))dxdy$$

- The eigenvalue decomposition of $\mathbf{K}$ and a linear combination of the *snapshots* to build $\{\phi_n(x, y)\}_{n=1}^{N}$

Parametric estimation

Iterative approach from the initial vector $\boldsymbol{\theta}_0$ and until convergence

- Find an approximative solution $\hat{u}(i, j, k)$
  - Simulate of the system of ODEs
  - Calculate the approximative solution

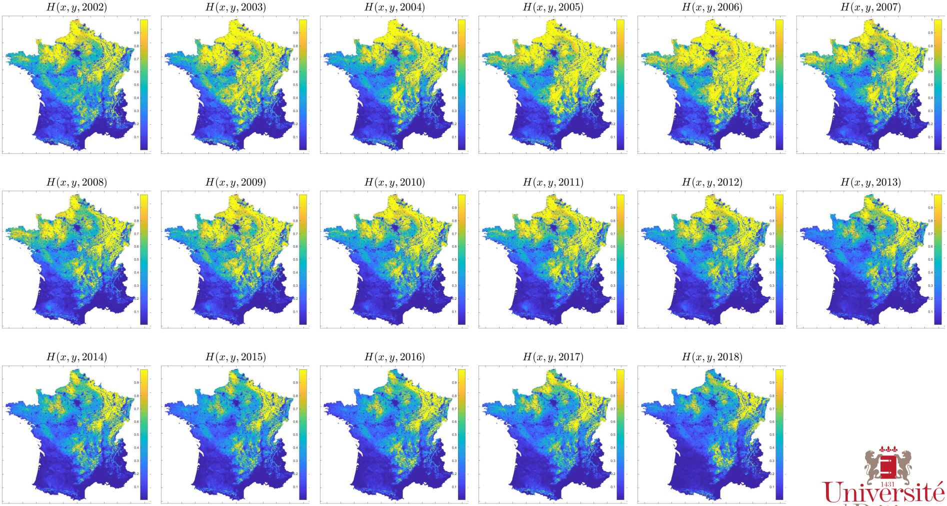
$$\hat{u}(x, y, t) = \sum_{n=1}^{N} a_n(t)\phi_n(x, y) + \bar{u}(x, y)$$

- Calculate a new parameter vector $\boldsymbol{\theta}_{iter+1}$ with the Levenberg-Marquardt algorithm

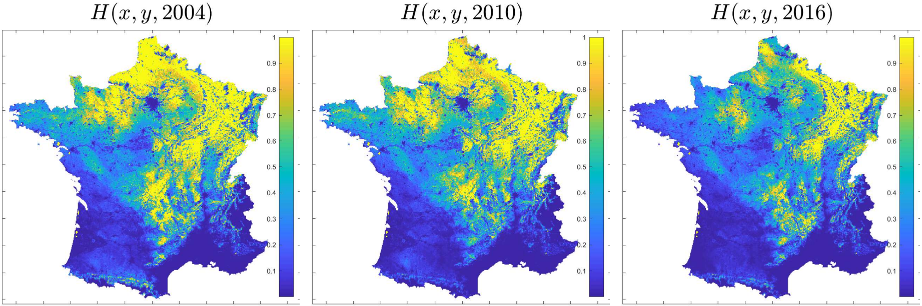
$$\boldsymbol{\theta}_{iter+1} = \boldsymbol{\theta}_{iter} - \left(\mathbf{J}''_{\theta\theta} + \mu \mathbf{I}_{N_\theta}\right)^{-1} \mathbf{J}'_\theta$$

with μ, a tuning parameter, $\mathbf{J}'_\theta$ and $\mathbf{J}''_{\theta\theta}$, the gradient and the pseudo-Hessian respectively

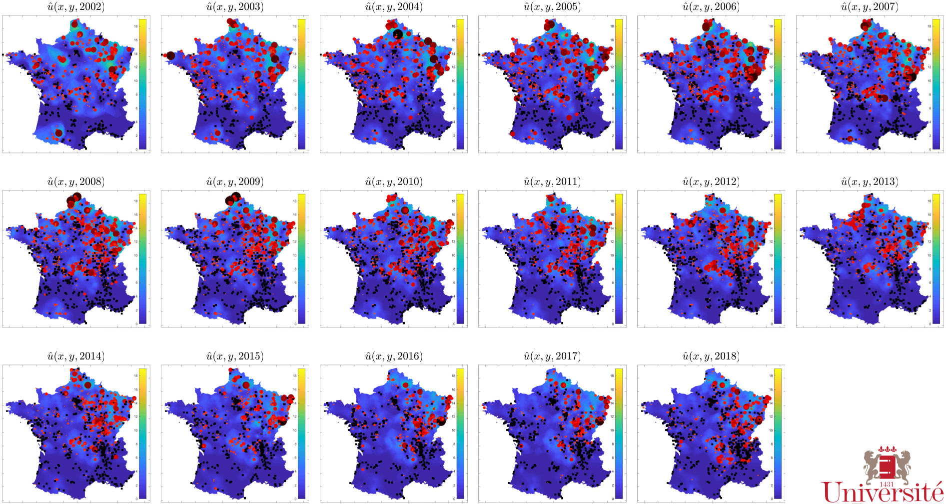
Scheduling variable $H(x, y, t)$



Scheduling variable $H(x, y, t)$

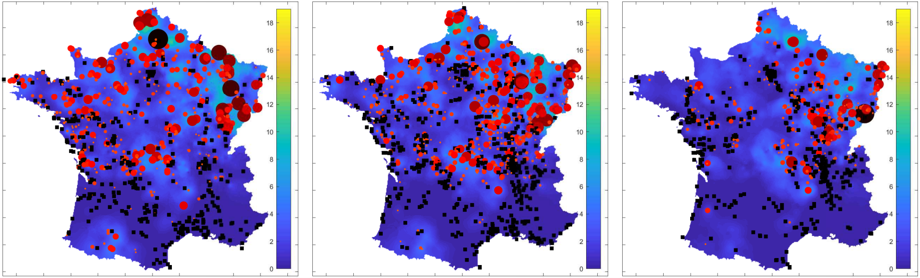


Output $\hat{u}$ of the estimated PDE model: Fitting of 44 %



Output $\hat{u}$ of the estimated PDE model: Fitting of 44 %

$\hat{u}(x, y, 2004)$ $\hat{u}(x, y, 2010)$ $\hat{u}(x, y, 2016)$



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Formulation problem

Lack of prior knowledge about the parameters for
the initialization $\boldsymbol{\theta}_0$ of the Levenberg-Marquardt algorithm

Solution

- Estimation a PDE with fixed parameters by using the partial moments approach
- Choice of the initial parameters of the PDE with variable parameters based on this estimation and the expertise of ecologists

Define the 3D partial moment with orders i, j and k of an arbitrary signal $f(x, y, t)$ belonging to a Hilbert space by

$$\mathcal{M}^{f}_{i,j,k}(X, Y, T) = \int_0^X \int_0^Y \int_0^T x^i y^j t^k f(x, y, t)\ dt\ dy\ dx$$

with i, j and $k \in \{\mathbb{N},\ -\}$

Example

Consider the system described by the PDE with fixed parameter

$$\frac{\partial u(x,y,t)}{\partial t} = D\ \mathrm{div}\left(\nabla u(x,y,t)\right) - w_x \frac{\partial u(x,y,t)}{\partial x} - w_y \frac{\partial u(x,y,t)}{\partial y} + \beta_1\ u(x,y,t) - \beta_2\ u^2(x,y,t)$$

Example

Apply the following calculation to the PDE

$$\int_0^X \int_0^{x_1} x_2^2 \int_0^Y \int_0^{y_1} y_2^2 \int_0^T t \quad PDE \quad dt \; dy_2 \; dy_1 \; dx_2 \; dx_1$$

Using the integration by parts and the Cauchy formula of repeated integrations leads to the linear regression

$$\mathcal{U}_1(X,Y,T) = \boldsymbol{\varphi}^\top(X,Y,T)\boldsymbol{\theta}_{LS}$$

with

$$\boldsymbol{\theta}_{LS} = \begin{bmatrix} D & w_x & w_y & \beta_1 & \beta_2 \end{bmatrix}^\top$$

$$\boldsymbol{\varphi}(X,Y,T) = \big[\; \mathcal{U}_2(X,Y,T) + \mathcal{U}_3(X,Y,T) \quad -\mathcal{U}_4(X,Y,T) \quad -\mathcal{U}_5(X,Y,T) \quad \mathcal{U}_6(X,Y,T) \quad -\mathcal{U}_7(X,Y,T) \; \big]^\top$$

New models of population dynamics

Initialisation by the partial moment approach

Example

$$
\begin{aligned}
\mathcal{U}_1 &= XYT\mathcal{M}^u_{2,2,-} - XT\mathcal{M}^u_{2,3,-} - YT\mathcal{M}^u_{3,2,-} + T\mathcal{M}^u_{3,3,-} \\
&\quad -XY\mathcal{M}^u_{2,2,0} + X\mathcal{M}^u_{2,3,0} + Y\mathcal{M}^u_{3,2,0} - \mathcal{M}^u_{3,3,0}, \\
\mathcal{U}_2 &= X^2Y\mathcal{M}^u_{-,2,1} - X^2\mathcal{M}^u_{-,3,1} + 2XY\mathcal{M}^u_{0,2,1} \\
&\quad -2X\mathcal{M}^u_{0,3,1} - 6Y\mathcal{M}^u_{1,2,1} + 6\mathcal{M}^u_{1,3,1}, \\
\mathcal{U}_3 &= XY^2\mathcal{M}^u_{2,-,1} - Y^2\mathcal{M}^u_{3,-,1} + 2XY\mathcal{M}^u_{2,0,1} \\
&\quad -2Y\mathcal{M}^u_{3,0,1} - 6X\mathcal{M}^u_{2,1,1} + 6\mathcal{M}^u_{3,1,1}, \\
\mathcal{U}_4 &= 3Y\mathcal{M}^u_{2,2,1} - 3\mathcal{M}^u_{2,3,1} - 2XY\mathcal{M}^u_{1,2,1} + 2X\mathcal{M}^u_{1,3,1}, \\
\mathcal{U}_5 &= 3X\mathcal{M}^u_{2,2,1} - 3\mathcal{M}^u_{3,2,1} - 2XY\mathcal{M}^u_{2,1,1} + 2Y\mathcal{M}^u_{3,1,1}, \\
\mathcal{U}_6 &= XY\mathcal{M}^u_{2,2,1} - Y\mathcal{M}^u_{3,2,1} - X\mathcal{M}^u_{2,3,1} + \mathcal{M}^u_{3,3,1}, \\
\mathcal{U}_7 &= XY\mathcal{M}^{u^2}_{2,2,1} - Y\mathcal{M}^{u^2}_{3,2,1} - X\mathcal{M}^{u^2}_{2,3,1} + \mathcal{M}^{u^2}_{3,3,1}
\end{aligned}
$$

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System

Consider the PDE with fixed parameter

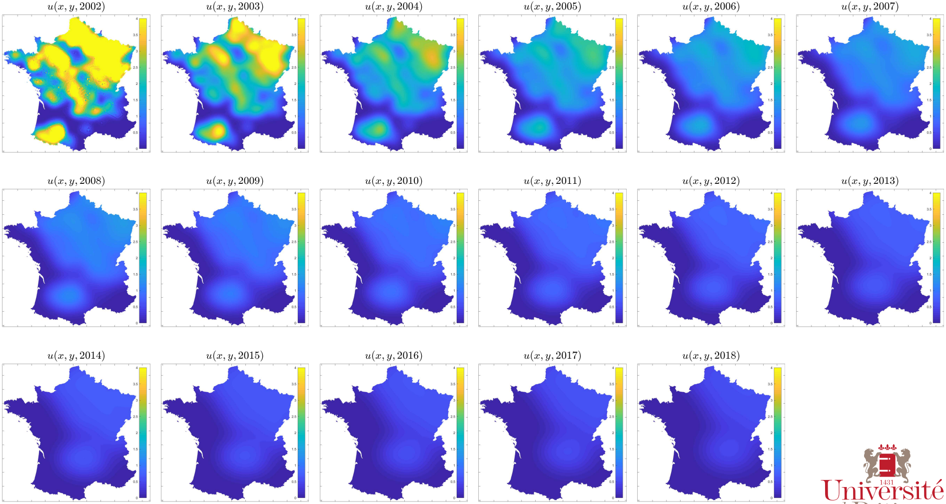
$$\frac{\partial u(x,y,t)}{\partial t} = D \,\mathrm{div}\left(\nabla u(x,y,t)\right) - w_x \frac{\partial u(x,y,t)}{\partial x} - w_y \frac{\partial u(x,y,t)}{\partial y} + \beta_1 \, u(x,y,t) - \beta_2 \, u^2(x,y,t)$$

with $D = 200$, $w_x = 18$, $w_y = 15$, $\beta_1 = 0.01$, $\beta_2 = 0.1$

Simulation protocol

- Finite difference method, Crank-Nicolson discretization scheme, splitting method for non-linearity
- Sampling $dx = 0.5$, $dy = 0.5$, $dt = 0.01$
- Domain $\Omega = [-500, 1500] \times [-500, 1500]$
- 17 time samples $t \in [2002, 2018]$

Simulated output $u(x, y, t)$ of the PDE system



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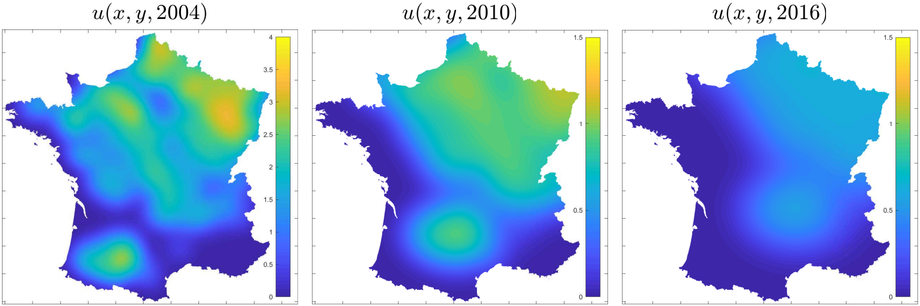

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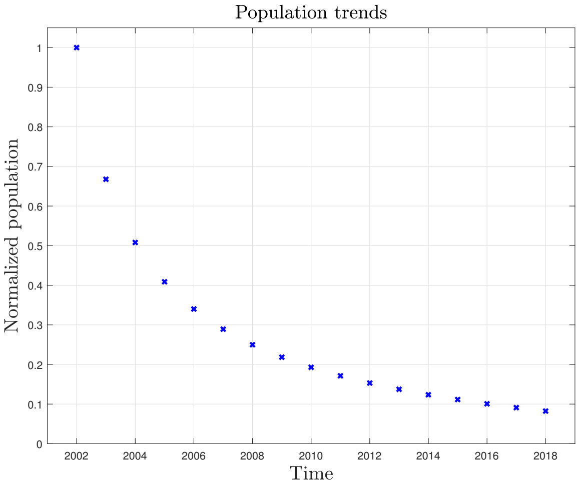


Estimation results using the partial moments approach

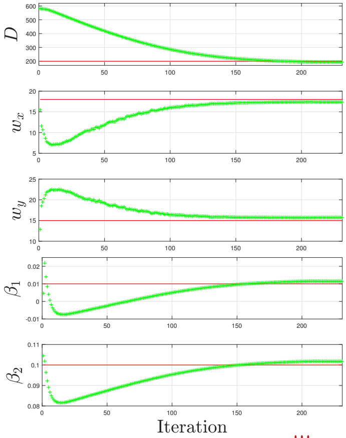
Sampling $dx = 50,\ dy = 50,\ dt = 1$

	True parameters	Estimated values	Relative errors (%)
D	200	581.0	190.5
w_x	18	15.5	13.9
w_y	15	12.8	14.4
β_1	0.01	0.005	53.8
β_2	0.1	0.104	4.5

Estimation result with the Levenberg-Marquardt algorithm

Sampling $dx = 5$, $dy = 5$, $dt = 1$

	True parameters	Estimated values	Relative errors (%)
D	200	194.1	3.0
w_x	18	17.4	3.5
w_y	15	15.7	4.8
β_1	0.01	0.012	14.8
β_2	0.1	0.102	1.8



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Conclusion

- PDEs with variable parameters, powerful models for the study of population dynamics
- Partial moments, a tool for initializing optimization algorithms

Perspectives

- Validate our modeling and parametric estimation tools on real applications at different geographical scales
- Determine confidence intervals using approaches such as Markov Chain Monte Carlo methods
- Predict the evolution of populations to evaluate public policies (Common Agricultural Policy, climate scenarios, infrastructures, etc.)
- Program algorithms in R in a free package