Stability of high frequency amplifiers via singularities of the Harmonic Transfer Function

Workshop ERNSI

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27 May 2021

Explaining the title

- Dynamical system with periodic solution
  $\Rightarrow$ Is it locally stable?

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- Characterization in the frequency domain.

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- Dynamical system with periodic solution
  $\Rightarrow$ Is it locally stable?
- Delay or 1-D Hyperbolic PDE.
- Characterization in the frequency domain.
- Nonlinear electric circuit containing transmission lines.

Summary

1 Introduction
- General context

2 Circuits without transmission lines

3 General circuits containing lossless transmission lines

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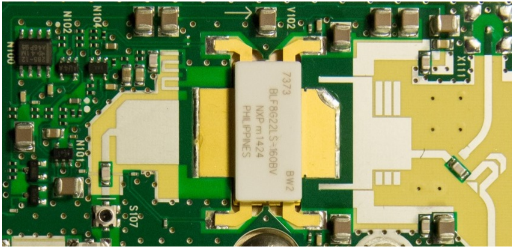
Inside

An amplifier is made of interconnected

1. resistors, inductors, capacitors,
2. diodes/transistors (active component),
3. lossless transmission lines inducing delays.

Forcing periodic input (large signal functioning)
▶▶ periodic solution in the amplifier.



Motivation

- Amplifiers at high frequency are ubiquitous (Cell phones, relays...). They need to be quick to design.
- Required Computer-assisted design (CAD) and simulation.
- time domain inefficient $\Rightarrow$ Need to work in frequency domain.
- "frequency simulation" tools give a reliable prediction of the periodic solution, but that solution might be unstable $\Leftarrow$ price to pay to work in the frequency domain.
- **Need** for a tool to predict stability/unstability from the frequency domain data.

Harmonic Balance: a numerical heuristic

The Harmonic Balance method, through Fourier development, Laplace transform and fixed point methods permits to :

- approximate a periodic solution of the circuit,
- linearize the (equations of) circuit around the periodic solution,
- compute the frequency response to a small periodic signal which disturbs the linearized circuit.

Harmonic balance method : Numerically implemented. (ADS, MWO and Spectre softwares)

KEYSIGHT TECHNOLOGIES NATIONAL INSTRUMENTS cādence

Stability of the periodic solution

►► Frequency response of the linearized system to a perturbation signal: input-output system

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►► Rational approximation tools used on the Fourier development of $G(i\omega, \cdot)$ (Harmonic Transfer Function)

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►► Implementation STAN tool software or see method in the paper [CSO^+18].

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Our focus in this talk:

- Structure of the **Harmonic Transfer Function**: analytic operator valued.
- its singularities, are they poles?
- links with local stability in the time domain.

1 Introduction
- General context

2 Circuits without transmission lines

3 General circuits containing lossless transmission lines

No transmissions line. Equations

Resistors, transistors, inductors, capacitors and forcing periodic signal

$$+$$

Kirshhoff's laws.

Assumption: Existence of a T-periodic solution.

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Linearization around periodic solution:

$$\frac{dx(t)}{dt} = A(t)x(t),$$

x: state of the circuit (currents or voltages),

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Kirshhoff's laws.

Assumption: Existence of a T-periodic solution.
Linearization around periodic solution:

$$\frac{dx(t)}{dt} = A(t)x(t),$$

x: state of the circuit (currents or voltages),
Paradigm: Exponential stability of the origin of the linearized equation implies local stability of the periodic solution.

No transmission lines. Time-invariant case

Dynamical system:

$$\frac{dx(t)}{dt} = Ax(t),$$

x: state of the circuit (current or voltage).

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Definition

The origin of the system is exponentially stable if there exists γ, K positive such that:

$$||x(t)|| \leq Ke^{-\gamma t}||x(0)||,$$

$x(\cdot)$ solution of the system.

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Theorem (classical)

Origin exp. stable if and only if the real part of the eigenvalues of A are strictly negative.

Time-invariant case

Input-output system :

$$\begin{cases} \frac{d}{dt}y(t) = Ay(t) + Bu(t) \\ z(t) = Cy(t) + Du(t)\ , t \geq 0. \end{cases}$$

Time-invariant case

Input-output system :

$$\begin{cases} \frac{d}{dt}y(t) = Ay(t) + Bu(t) \\ z(t) = Cy(t) + Du(t) \ , t \geq 0. \end{cases}$$

Transfer function : $H(s) := C(sI_n - A)^{-1}B + D$.
It satisfies the property: $Z(s) = H(s)U(s)$,
$Z(\cdot)$, $U(\cdot)$ Laplace transform of $z(\cdot)$, $u(\cdot)$.

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Theorem (classical)

If the system is observable and controllable, the singularities of the transfer function are poles and are the eigenvalues of A. Therefore the system is exponentially stable if and only if $H(s)$ analytic in the closed right half plane.

Periodic case: Nonlinear

$$\frac{dx(t)}{dt} = A(t)x(t),\ T > 0 \text{ periodic.}$$

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$\Rightarrow$ Eigenvalues of $A(t)$ do not give the exponential stability of the origin anymore.

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Periodic ODE $\Rightarrow$ Subsumed under **Floquet Theory**.

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We introduce the fundamental solution $X(t, t_0)$ which satisfies :

$$\begin{aligned} \frac{d}{dt}X(t, t_0) &= A(t)X(t, t_0), \; t \geq t_0, \\ X(t_0, t_0) &= Id, \; \text{for all } t_0 \in \mathbb{R}. \end{aligned}$$

Definition (monodromy operators)

The monodromy operators of the system are the operator $X(T + t_0, t_0)$ for t_0 real.

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Theorem (classical)

Origin exp. stable if and only if the spectrum of the monodromy $X(T, 0)$ operator is strictly included in the unit disk.

Periodic case: Input-output system

Input-output system:

$$\begin{cases} \frac{d}{dt}y(t) = A(t)y(t) + B(t)u(t) \\ z(t) = C(t)y(t) + D(t)u(t), \;\; t \geq 0, \end{cases}$$

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Fourier series:

$$\begin{cases} \frac{d}{dt}y(t) = \sum\limits_{k=-\infty}^{\infty} a_k e^{i\omega_0 kt} y(t) + \sum\limits_{k=-\infty}^{\infty} b_k e^{i\omega_0 kt} u(t), \\ z(t) = \sum\limits_{k=-\infty}^{\infty} c_k e^{i\omega_0 kt} y(t) + \sum\limits_{k=-\infty}^{\infty} d_k e^{i\omega_0 kt} u(t), \end{cases}$$

$\omega_0 := 2\pi/T$.

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$\omega_0 := 2\pi/T$.
Fourier series + Laplace transform $\Rightarrow$ Infinite time-invariant system.

Harmonic Transfer Function

$$Z(s) = \quad ? \quad \begin{pmatrix} \vdots \\ U(s+i\omega_0) \\ U(s) \\ U(s-i\omega_0) \\ \vdots \end{pmatrix}$$

Harmonic Transfer Function

There exists $H(\cdot)$ infinite matrix such that:

$$\begin{pmatrix} \vdots \\ Z(s+i\omega_0) \\ Z(s) \\ Z(s-i\omega_0) \\ \vdots \end{pmatrix} = H(s) \begin{pmatrix} \vdots \\ U(s+i\omega_0) \\ U(s) \\ U(s-i\omega_0) \\ \vdots \end{pmatrix}$$

Definition

The infinite matrix $H(s) := L_C[D_{\omega_0}(s) - L_A]^{-1} + L_D$ is called the **Harmonic Transfer Function** (HTF) matrix. (Wereley)

Where $D_{\omega_0}(s) := Diag(\cdots, s+i\omega_0, s, s-i\omega_0, \cdots)$ and $L_A := (a_{j-i})_{i,j\in\mathbb{Z}}$, $L_B := (b_{j-i})_{i,j\in\mathbb{Z}}$, $L_C := (c_{j-i})_{i,j\in\mathbb{Z}}$ and $L_D := (d_{j-i})_{i,j\in\mathbb{Z}}$.

HTF, singularities?

- HTF defines an operator valued analytic map (values: continuous ops $l^2(\mathbb{Z}) \to l^2(\mathbb{Z})$)

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►► Can be a singularity of the HTF, but what is the link with the singularities of the entries of the HTF?

Singularities HTF versus singularities of its entries

Let $H(s)$ an operator valued analytic map (values: continuous ops $l^2(\mathbb{Z}) \to l^2(\mathbb{Z})$) on an open set S.

- $s_0 \in \overline{S}$ is a singularity of $H(\cdot)$ if $H(\cdot)$ does not admit an analytic continuation in s_0. In particular if $H(s_0)$ is not a bounded operator from l^2 to l^2 then it is a singularity.

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- However each coefficient of the infinite matrix can be defined in s_0.

As example s_0 is an essential singularity:

$$\lim_{n \to +\infty} \|H(s_0)x_n\| = +\infty,$$

where $||x_n|| = 1$ and all the coefficients of $H(s_0)$ exists and are finite.

Stability and HTF

Dwelling on **Floquet Theory**, we have:

Theorem (After Suarez and Quéré 02 or Louarroudi 14 for example)

Each element of the HTF is a meromorphic function. And its poles are included (possibly strictly) in the set of the complex numbers of the form $\frac{ln(\zeta)+2i\pi k}{T}$ *for all* $k \in \mathbb{Z}$ *and where* ζ *is an eigenvalue of the monodromy operator.*

Theorem (classical)

Assume the system is observable and controllable, then for all ζ *an eigenvalue of the monodromy operator there exists* k *integer such that* $H(s)$*, and one of its entries, has a pole in* $\frac{ln(\zeta)+2i\pi k}{T}$*.*
Therefore origin exp. stable if and only if $H(s)$ *analytic in the closed right half plane.*

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Linearization around periodic solution:

$$\begin{cases} \frac{dx(t)}{dt} = A_1(t)x(t) + \sum_{i=0}^{N} B_{1,i}(t)y(t-\tau_i) \\ y(t) = \sum_{i=1}^{N} B_{2,i}(t)y(t-\tau_i) + A_2(t)x(t), \ t \geq t_0. \end{cases}$$

square integrable and continuous solutions

For electronic applications: continuous solution and C^0 stability are needed.

However Fourier development of the system: L^2 solutions needed.

Anyway: L^2 stability is equivalent to the C^0 stability.

General linear system, T-periodic

$$\begin{cases} \frac{dx(t)}{dt} = A_1(t)x(t) + \sum\limits_{i=0}^{N} B_{1,i}(t)y(t-\tau_i) \\ y(t) = \sum\limits_{i=1}^{N} B_{2,i}(t)y(t-\tau_i) + A_2(t)x(t), \ \ t \geq t_0, \end{cases}$$

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- State space: $\mathcal{Y}L^2 := \mathbb{R}^n \times L^2([-\tau_N, 0], \mathbb{R}^k)$.

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- Solution operator $U(t, t_0) : \mathcal{Y}L^2 \to \mathcal{Y}L^2$

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- Monodromy operator $U(T, 0)$

General linear system, T-periodic

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- Solution operator $U(t, t_0) : \mathcal{Y}L^2 \to \mathcal{Y}L^2$
- Monodromy operator $U(T, 0)$
- $$\left.\begin{array}{c} L^2 \text{ exponential} \\ \textit{stability} \end{array}\right\} \Leftrightarrow \left\{\begin{array}{c} Sp(U(T,0)) \text{ included in} \\ \text{disc of radius } r < 1 \end{array}\right.$$

General linear system, T-periodic

$$
\begin{cases}
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Behaviour at high frequency

High frequency limit system :

$$\begin{cases} x(t) = 0 \\ y(t) = \sum_{i=1}^{N} B_{2,i}(t) y(t - \tau_i), \ t \geq t_0, \end{cases}$$

- State space: $\mathcal{Y}\widetilde{L}^2 := \{0_n\} \times L^2([-\tau_N, 0], \mathbb{R}^k)$.
- Solution operator $V(t, t_0) : \mathcal{Y}\tilde{L}^2 \to \mathcal{Y}\tilde{L}^2$.
- Monodromy operator $V(T, 0)$.
- $$\left. \begin{array}{c} L^2 \text{ exponential} \\ \textit{stability} \end{array} \right\} \Leftrightarrow \left\{ \begin{array}{l} Sp(V(T,0)) \text{ included in} \\ \text{disc of radius } r < 1 \end{array} \right.$$

Compact perturbation

Lemma

We have :

$$U(t, t_0) = V(t, t_0)P + K(t, t_0), \ \ t \geq t_0$$

with $K(t, t_0)$ *compact operator* $\mathcal{Y}L^2 \to \mathcal{Y}L^2$ *for all* t, t_0 *and* P *the canonical projection* $\mathcal{Y}L^2 \to \mathcal{Y}\tilde{L}^2$.

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with $K(t,t_0)$ *compact operator* $\mathcal{Y}L^2 \to \mathcal{Y}L^2$ *for all* t, t_0 *and* P *the canonical projection* $\mathcal{Y}L^2 \to \mathcal{Y}\tilde{L}^2$.

Theorem (classical)

A compact perturbation does not change the essential spectrum and it can add only point spectrum, i.e. eigenvalues.

Theorem

If the high frequency system is L^2 ***exponentially stable*** *then the monodromy operator* $U(T, 0)$ *possesses at most a finite number of eigenvalues* $\zeta_1, \cdots, \zeta_n$ *outside a disk of a radius strictly less than* 1*.*

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(BFLP21)$\Rightarrow$ L^2 exp. stability of the high frequency system. It is a reasonable assumption for electric circuits.

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$\Rightarrow$ High frequency system dissipates energy
(BFLP21)$\Rightarrow$ L^2 exp. stability of the high frequency system. It is a reasonable assumption for electric circuits.
$\Rightarrow$ Instability of the general linear system is governed just by a finite number of eigenvalues.

Input-Output system

$$\begin{cases} \frac{dx(t)}{dt} = A_1(t)x(t) + \sum_{i=0}^{N} B_i^1(t)y(t-\tau_i) + C_1(t)u(t) \\ y(t) = \sum_{i=1}^{N} B_i^2(t)y(t-\tau_i) + A_2(t)x(t) + C_2(t)u(t) \\ z(t) = \sum_{i=1}^{N} B_i^3(t)y(t-\tau_i) + A_3(t)x(t) + C_3(t)u(t), \ \ t \geq 0, \end{cases}$$

- $x(t), y(t), z(t) = 0$ for $t < 0$,
- Input $u \in L^2_{loc}([0,+\infty),\mathbb{R})$ current perturbation, output z the voltage,
- All coefficients are $T - periodic$.

Harmonic Transfer Function

$$\begin{pmatrix} \vdots \\ Z(s+i\omega_0) \\ Z(s) \\ Z(s-i\omega_0) \\ \vdots \end{pmatrix} = H(s) \begin{pmatrix} \vdots \\ U(s+i\omega_0) \\ U(s) \\ U(s-i\omega_0) \\ \vdots \end{pmatrix}$$

- HTF is an operator valued analytic map (values: continuous ops $l^2(\mathbb{Z}) \to l^2(\mathbb{Z})$)
- its entries are complex valued analytic maps

Structure of the Harmonic Transfer Function

Define $z_{j,k} = \frac{ln(\zeta_j)+2ik\pi}{T}$ for j in $\{1...n\}$, k in $\mathbb{Z}$.
Fundamental structure HTF for high dissipative amplifiers

Theorem

In the half space $\{s \in \mathbb{C}, \Re(s) \geq \gamma\}$ for some $\gamma < 0$,

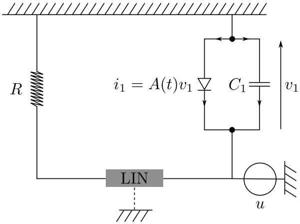
- *H is a meromorphic operator $l^2(\mathbb{Z}) \to l^2(\mathbb{Z})$ whose poles are contained in $\{z_{j,k}, j \in \{1...n\}, k \in \mathbb{Z}\}$.*
- *Under observability/controllability assumptions, for all j, there is at least a k such that $z_{j,k}$ is one pole of H, and also a pole of one entries of the HTF.*

If no entry of the HTF has poles in right half plane, we have the exponential C^0 stability.

Singularities of the HTF in the left right half plane?

- High frequency dissipative circuit $\Rightarrow$ HTF meromorphic right half plane.
- Rational tools sometimes performes rational approximation on all the complex plane.
- Needs to know if there is just poles in the left half plane?

Simple example (Brayton 1976)



Input: u

output: $z = v_1$

Theorem

If $T/\tau_1 \notin \mathbb{Q}$ and the system is observable, all points of a certain vertical line in the left-half plane are essential singularities of the HTF, as an operator valued analytic map.

But are they singularities for some entries of the HTF ?

Thank You!